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REDUCIBILITY OF EUCLIDEAN IMMERSIONS OF LOW CODIMENSION

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1. Introduction

By a theorem of Kobayashi, the holonomy algebra of a compact D-dimensional Riemannian manifold M, isometrically immersed in Euclidean space $\boldsymbol{R}^{D+1}$, is the full orthogonal algebra (M is not reducible, therefore). Suppose M is a reducible compact D-dimensional manifold having an isometric immersion ϕ in $\boldsymbol{R}^{D+2}$. A theorem of R. L. Bishop gives the holonomy algebra of M at m to be the sum $o(K) + o(D-K)$ of two orthogonal algebras acting on complementary orthogonal subspaces of the tangent space M_m. We show (Theorem 8.2) that, at least when ϕ is one-one, ϕ is in fact the product of two immersions of hypersurfaces, with an exception occurring in the case $K = 1$ or $D - 1$.

In § 9, certain Euclidean immersions are shown to be cylindrical. The following result, for example, follows from the codimension one case and a well-known theorem of Hartman and Nirenberg: If a complete D-dimensional manifold M has an isometric immersion ϕ in $\boldsymbol{R}^{D+1}$, then M is a Riemannian product $M_1 \times \boldsymbol{R}^{D-K}$, where the restricted holonomy group of M_1 acts irreducibly, and ϕ is $(D-K)$-cylindrical.

Throughout, M indicates a connected Riemannian manifold, and all structures are C^∞.

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2. Isometric immersions

Some basic material concerning an isometric immersion $\phi: M \to \overline{M}$ is outlined here, largely to establish notation.

Let K and $\overline{K}$ be the Riemannian tangent bundles of M and $\overline{M}$ respectively. K is identified through the tangent map $d\phi$ with a metric sub-bundle of $\overline{K}|M$; $K^\perp$ will be the sub-bundle with complementary orthogonal fiber over each m (we write: $M_m + M_m{}^\perp = \overline{M}_m$). Letting $\mathfrak{F}$ be the algebra of smooth

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functions on M, we have the $\mathfrak{F}$-modules $\bar{\mathfrak{X}}$, $\mathfrak{X}$ and $\mathfrak{X}^\perp$, consisting of the smooth sections of $\bar{K}|M$, K and $K^\perp$ respectively. $\bar{\mathfrak{X}}$ is the direct sum of $\mathfrak{X}$ and $\mathfrak{X}^\perp$, and P ($P^\perp$) will be the corresponding projection of $\bar{\mathfrak{X}}$ onto $\mathfrak{X}$ ($\mathfrak{X}^\perp$).

Consider the Riemannian connection $\bar{\nabla}$ on $\bar{K}$: or, rather, the restriction connection $\bar{\nabla}: \mathfrak{X} \times \bar{\mathfrak{X}} \to \bar{\mathfrak{X}}$ on $\bar{K}|M$. Then the Riemannian connection on K is $\nabla: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}: (X, Y) \to \nabla_X Y$ where $\nabla_X Y = P\bar{\nabla}_X Y$. We write

$$(1) \qquad T_X Y = P^\perp \bar{\nabla}_X Y, \qquad X, Y \text{ in } \mathfrak{X}.$$

On $K^\perp$, $P^\perp$ induces the connection

$$\nabla^\perp: \mathfrak{X} \times \mathfrak{X}^\perp \to \mathfrak{X}^\perp: (X, Z) \to \nabla^\perp{}_X Z,$$

where $\nabla^\perp{}_X Z = P^\perp \bar{\nabla}_X Z$. Here, we write

$$(2) \qquad T_X Z = P\bar{\nabla}_X Z, \qquad X \text{ in } \mathfrak{X},\ Z \text{ in } \mathfrak{X}^\perp.$$

Let $R: \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be the curvature tensor of M (that is, of the Riemannian connection ∇ on K). By definition, R assigns to every (X, Y) in $\mathfrak{X} \times \mathfrak{X}$ the mapping $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ of $\mathfrak{X}$ into $\mathfrak{X}$. R is trilinear over $\mathfrak{F}$, hence determines, for m in M and x, y in M_m, the skew-symmetric curvature transformation R_{xy} of M_m into itself. Let $R^\perp: \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}^\perp \to \mathfrak{X}^\perp$ be the curvature tensor of $\nabla^\perp$ on $K^\perp$, similarly defined by the equation $R^\perp{}_{XY} = \nabla^\perp{}_{[X,Y]} - [\nabla^\perp{}_X, \nabla^\perp{}_Y]$ and determining the "normal curvature" transformations $R^\perp{}_{xy}$ of $M_m{}^\perp$.

Then by the Ambrose-Singer holonomy theorem, the holonomy algebra at m of M is spanned by the parallel translates to m of the curvature transformations R_{xy} at all of the points of M; and the holonomy algebra at m of the connection $\nabla^\perp$ on $K^\perp$ is similarly determined by the transformations $R^\perp{}_{xy}$.

3. The difference operator T

(1) and (2) define a bilinear mapping T over $\mathfrak{F}$ of $\mathfrak{X} \times \bar{\mathfrak{X}}$ into $\bar{\mathfrak{X}}$. At each m, the resulting bilinear transformation T of $M_m \times \bar{M}_m$ into $\bar{M}_m$ satisfies, for x, y in M_m and z in $M_m{}^\perp$: $T_x y \in M_m{}^\perp$; $T_x z \in M_m$; $T_x y = T_y x$; and $\langle T_x y, z\rangle = -\langle T_x z, y\rangle$. Here, $\langle\ ,\ \rangle$ is the inner product on $\bar{M}_m$. Clearly, the action of T_x on $\bar{M}_m$ is determined by its action on M_m. A more detailed discussion of the tensor T may be found in [4].

The following two results are easily verified:

3.1. Lemma. *Let $\bar{R}$ be the curvature tensor of $\bar{M}$. Then for x, y in M_m,*

$$R_{xy} = P\bar{R}_{xy} + [T_x, T_y],$$
$$R^\perp{}_{xy} = P^\perp \bar{R}_{xy} + [T_x, T_y].$$

3.2. Lemma. *M is totally geodesic in $\bar{M}$ under ϕ (that is, geodesics of M are geodesics of $\bar{M}$, under ϕ) if and only if T is zero everywhere.*

The second fundamental form transformations S_z: $M_m \to M_m$ of ϕ at m are defined by

$$S_z x = T_x z\,, \qquad z \text{ in } M_m{}^\perp,\ x \text{ in } M_m\,. \tag{3}$$

Since $\langle S_z x, y\rangle = -\langle T_x y, z\rangle$, and $T_x y = T_y x$, the S_z are symmetric.

Finally, suppose $M_1 \to M_2 \to M_3$ is a chain of isometric immersions, with T_{ij} the difference operator of the immersion of M_i in M_j. Then for x, y in $(M_1)_m$, we have

$$(T_{13})_x y = (T_{12})_x y + (T_{23})_x y\,. \tag{4}$$

4. Euclidean immersions

We apply Lemma 3.1 to an isometric immersion ϕ: $M \to R^{D+E}$ of a D-dimensional manifold M in $(D+E)$-dimensional Euclidean space. At any m in M, the relation $R^\perp{}_{xy} = [T_x, T_y]$ holds on $M_m{}^\perp$:

4.1. Lemma [2, p. 230]. *The normal curvature tensor $R^\perp$ of ϕ is zero at m if and only if the second fundamental form transformations of ϕ at m are simultaneously diagonalizable.*

Proof. Lemma 3.1 and (3) give $\langle R^\perp{}_{xy} z, z'\rangle = \langle x, [S_z, S_{z'}]y\rangle$, for z, z' in $M_m{}^\perp$. A set of symmetric linear transformations of M_m is commutative if and only if the transformations simultaneously have diagonal form with respect to some orthogonal basis. q.e.d.

In [1], the formula $R_{xy} = [T_x, T_y]$ is given a useful expression in terms of the second fundamental form transformations. The orthogonal algebra $o(M_m)$ (skew-symmetric endomorphisms of M_m) and the space $M_m{}^2$ of Grassmann bivectors on M_m are identified, according to the rule $xy(w) = \langle x, w\rangle y - \langle y, w\rangle x$. For later reference, we include the formula for Lie product in $M_m{}^2$:

$$\begin{aligned}[xy, vw] = &\langle x, v\rangle yw + \langle y, w\rangle xv \\ &- \langle x, w\rangle yv - \langle y, v\rangle xw\,.\end{aligned} \tag{5}$$

If $z_{D+1}, \cdots, z_{D+E}$ are a normal frame at m (an orthonormal basis of $M_m{}^\perp$), setting $S_a = S_{z_a}$ $(D+1 \le a \le D+E)$ and regarding R_{xy} as a bivector gives:

4.2. Lemma. $\qquad R_{xy} = \sum_a (S_a x)(S_a y)\,.$

5. Relative nullity

The relative nullity index ν of ϕ: $M \to \bar{M}$ is the integer-valued function on M defined as follows [3]: let the relative nullity space $\mathscr{R}(m)$ of ϕ at m consist of all x in M_m satisfying $T_x = 0$, where T is the difference operator of ϕ, and let $\nu(m)$ be the dimension of $\mathscr{R}(m)$.

In the case $\overline{M} = \boldsymbol{R}^{D+E}$ of interest here, we have:

5.1. Lemma [10], [5]. *Suppose the relative nullity index of the isometric immersion $\phi\colon M \to \boldsymbol{R}^{D+E}$ is constant on M. Then the relative nullity distribution $\mathscr{R}$ on M is smooth and integrable, its leaves are totally geodesic in $\boldsymbol{R}^{D+E}$, and the tangent planes to M are Euclidean parallel on each leaf.*

Proof. If $T_X = T_Y = 0$ for X, Y in $\mathfrak{X}$, then $T_{\nabla_X Y} = 0$; indeed, for any vector field W in $\mathfrak{X}$,

$$\begin{aligned} T_W \nabla_X Y &= P^{\perp}\bar{\nabla}_W \bar{\nabla}_X Y = P^{\perp}\bar{\nabla}_X \bar{\nabla}_W Y - P^{\perp}\bar{\nabla}_{[X,W]} Y \\ &= T_X \nabla_W Y - T_{[X,W]} Y = 0 \,. \end{aligned}$$

Differentiability of $\mathscr{R}$ follows from its definition. By the remark above, $\nabla_X Y$ is nullity whenever X and Y are. Then by the Frobenius theorem, since $[X, Y] = \nabla_X Y - \nabla_Y X$, $\mathscr{R}$ is integrable; and by Lemma 3.2, each leaf L of $\mathscr{R}$ is totally geodesic in M. Indeed, since $T_{\mathscr{R}(m)}\mathscr{R}(m) = 0$ everywhere, (4) implies that L is totally geodesic in $\boldsymbol{R}^{D+E}$ under $\phi \mid L$. The final statement is immediate from the defining relation, $T_{\mathscr{R}(m)} M_m = 0$ for all m.

6. Relative nullity foliations

Let M be a manifold having an isometric immersion ϕ in $\boldsymbol{R}^{D+E}$. Suppose N is an open subset of M on which the relative nullity index of ϕ is constant, and let $\mathscr{R}$ be the relative nullity distribution on N. Let m be a point of N, L the leaf through m of $\mathscr{R}$, and m^* any point of the closure of L in M.

For x in M_m or $M_m{}^{\perp}$, let x^* be the Euclidean parallel translate of x to m^*. By Lemma 5.1, we have $M_{m^*} = (M_m)^*$.

6.1. Lemma. *There is an isomorphism $U = U(m^*)$ of M_m onto M_{m^*} satisfying, for all x and y in M_m,*

$$(T_x y)^* = T_{Ux} y^* \,, \tag{6}$$

where T is the difference operator of ϕ.

Proof. We will define a transformation $U(m^*)\colon M_m \to M_{m^*}$ satisfying (6), and non-singular on $\mathscr{R}(m)$. Then $U(m^*)$ is non-singular; by (6), $Ux = 0$ implies $T_x = 0$, hence that x is in $\mathscr{R}(m)$.

On $\mathscr{R}(m)$, let $U(m^*)$ be Euclidean parallel translation to m^*. Since the translate of $\mathscr{R}(m)$ is $\mathscr{R}(m^*)$ if m^* lies in L, and hence is nullity if m^* is any limit point of L, (6) is satisfied for x in $\mathscr{R}(m)$. We now define $U(m^*)$ on $\mathscr{R}(m)^{\perp} \cap M_m$:

By Lemma 5.1, if l is the dimension of $\mathscr{R}$, the leaf L of $\mathscr{R}$ lies in a Euclidean l-plane under ϕ. For n in L, let $P(n)$ denote the complementary orthogonal plane through $\phi(n)$. Each point of L has a coordinate neighbourhood C

in N, for which the leaves of $\mathscr{R}|C$ and the slices through C by the planes $P(n)$ are complementary coordinate slices.

Suppose m^* is in L. Joining m and m^* in L, with a path covered by coordinate neighbourhoods C, we see that the leaves of $\mathscr{R}$ establish a diffeomorphism from a neighbourhood of m in the slice through M by $P(m)$, onto a neighbourhood of m^* in the slice through M by $P(m^*)$. Let $U(m^*)$ be the corresponding tangent map at m, from $\mathscr{R}(m)^\perp \cap M_m$ onto $\mathscr{R}(m^*)^\perp \cap M_{m^*}$. (6) follows from the definition of T and the fact that tangent planes of M are Euclidean parallel on each leaf of $\mathscr{R}$.

Now suppose m^* is in the closure of L. If the ϕ-image of each leaf of $\mathscr{R}$ is extended to a complete I-plane in $\boldsymbol{R}^{D+E}$, these planes establish by intersection with $P(m^*)$, a differentiable mapping into $P(m^*)$ from a neighbourhood of m in the slice through M by $P(m)$. If m^* is in L, the corresponding tangent map at m, from $\mathscr{R}(m)^\perp \cap M_m$ into $\mathscr{R}(m^*)^\perp$, agrees with $U(m^*)$. This shows that $U(m^*)$ is uniquely defined for any m^* in L, and that if m^* is not in L, then U has a continuous extension to m^*. q.e.d.

This lemma may also be deduced from the paper [5] by Philip Hartman. Then we have the following theorem of Hartman (proved in an original version by Barrett O'Neill [10], under the additional assumption that M be flat):

6.2. Theorem [5]. *Suppose M is a manifold with isometric immersion ϕ in Euclidean space. Let N be an open subset of M on which the relative nullity index of ϕ is constant, say $\nu(N) = I$, and let $\mathscr{R}$ be the relative nullity distribution on N. Then ν takes constant value I on the closure of each leaf of $\mathscr{R}$.*

In particular, let N be the open subset having minimum relative nullity. Then each leaf of $\mathscr{R}$ is closed in M; if M is complete, each leaf of $\mathscr{R}$ is complete.

Proof. If m is in a leaf L of $\mathscr{R}$, and m^* is in the closure of L, then by Lemma 6.1 the relative nullity space of ϕ at m^* is $\mathscr{R}(m)^*$. This verifies the first claim, which, together with the fact that the leaves of $\mathscr{R}$ are totally geodesic in $\boldsymbol{R}^{D+E}$, implies the rest. q.e.d.

We will need a generalization of Lemma 6.1 to the case where N is not open in M. Again, let M have an isometric immersion ϕ in $\boldsymbol{R}^{D+E}$. Suppose N is a Riemannian submanifold of M such that (i) the relative nullity index of the isometric immersion $\phi|N$ of N in $\boldsymbol{R}^{D+E}$ is constant, and (ii) at every n in N, the relative nullity space of $\phi|N$ is the intersection of N_n and the relative nullity space of ϕ.

Let $\mathscr{R}$ be the relative nullity distribution of $\phi|N$ on N. Choose m in N, and choose m^* in the closure in M, of the leaf L of $\mathscr{R}$ through m; let x^* be the Euclidean parallel translate of x from m to m^*. Then (ii) implies $M_{m^*} = (M_m)^*$. Of course, if m^* is in L, Lemma 5.1 implies $N_{m^*} = (N_m)^*$.

6.3. Lemma. *There is an isomorphism $U = U(m^*)$ of N_m onto $(N_m)^*$ satisfying, for all x in N_m and y in M_m,*

$$(T_x y)^* = T_{Ux} y^* ,$$

where T is the difference operator of ϕ.

Proof. Condition (ii) is sufficient to allow a proof exactly corresponding to the proof of Lemma 6.1.

7. Reducibility

We consider a reducible Riemannian manifold, that is, a manifold M for which the holonomy group H_m has a nontrivial invariant subspace $\mathscr{A}(m)$ in M_m. $\mathscr{A}(m)$ extends to a self-parallel (hence integrable) distribution $\mathscr{A}$ on M. Let $A(n)$ be the leaf through n of $\mathscr{A}$, and $B(n)$ the leaf through n of $\mathscr{B} = \mathscr{A}^\perp$. These leaves are totally geodesic in M, and are complete if M is complete.

Every point m in M has a neighbourhood N for which each leaf $A'(n)$ of $\mathscr{A}|N$ intersects each leaf $B'(n)$ of $\mathscr{B}|N$ exactly once, and for which the mapping $n \to (B'(n) \cap A'(m), A'(n) \cap B'(m))$ is an isometry of N and $A'(m) \times B'(m)$.

If each leaf of $\mathscr{A}$ intersects each leaf of $\mathscr{B}$ exactly once, it follows that the mapping $n \to (B(n) \cap A(m), A(n) \cap B(m))$ is an isometry of M and $A(m) \times B(m)$. The de Rham product theorem states that when M is simply connected and complete, the leaves of $\mathscr{A}$ and $\mathscr{B}$ have this unique intersection property. When M is complete, the de Rham theorem applied to the (complete) simply connected Riemannian covering manifold of M implies that each leaf of $\mathscr{A}$ intersects each leaf of $\mathscr{B}$ at least once.

A more thorough discussion may be found in [7] (especially pp. 179–193, p. 162).

8. Immersions of codimension two

Suppose M is a compact D-dimensional manifold isometrically immersed in $\boldsymbol{R}^{D+2}$. We have the following theorem of Bishop:

8.1. Theorem [1]. *If* $D \neq 4$, *the holonomy algebra* h_m *of* M *at* m *has the form* $o(U) + o(U^\perp)$ *where* U *is a* K*-dimensional subspace of* M_m $(0 \leq K \leq D$; *for convenience, we may say instead that* h_m *has the form* $o(K) + o(D - K))$. *If* $D = 4$, *the unitary algebra of some complex structure on* M_m *is also a possibility.*

For any positive integers D and $K \leq D$, examples may be found of Euclidean immersions of codimension two yielding holonomy algebra $o(K) + o(D - K)$, as above. Indeed, two isometric immersions $\phi_1: A \to \boldsymbol{R}^{K+1}$, $\phi_2: B \to \boldsymbol{R}^{D-K+1}$ of hypersurfaces give rise to an isometric immersion $\phi_1 \times \phi_2$ of the Riemannian product $A \times B$ in $\boldsymbol{R}^{D+2}$: for a in A and b in B, let $(\phi_1 \times \phi_2)(a, b) = (\phi_1 a, \phi_2 b)$. If A and B are compact, $A \times B$ has the required holonomy.

8.2. Theorem. *Let M be a reducible compact manifold of dimension $D > 2$, having an isometric immersion ϕ in $\boldsymbol{R}^{D+2}$. By Theorem* 8.1, *the holonomy algebra of M has the form $o(K) + o(D-K)$ where $1 \leq K \leq D-1$. Suppose (M^*, π) is the simply connected Riemannian covering of M. Then* (i) *if $2 \leq K \leq D-2$, $\phi \circ \pi \colon M^* \to \boldsymbol{R}^{D+2}$ is the product of two Euclidean immersions of hypersurfaces*; (ii) *if $K = 1$ or $D-1$, the same conclusion holds under the additional assumption that the normal curvature tensor of ϕ be zero.*

8.3. Corollary. *In case ϕ is one-one, the statement "ϕ is the product of two Euclidean immersions of hypersurfaces" may be substituted in Theorem* 8.2.

Proof of Corollary 8.3. By assumption, M carries self-parallel distributions $\mathscr{A}$ and $\mathscr{B} = \mathscr{A}^{\perp}$ of dimensions K and $D-K$ respectively. Theorem 8.2 states that, under the hypotheses of (i) or (ii), the leaves $A(m)$ of $\mathscr{A}$ lie in parallel Euclidean $(K+1)$-planes under ϕ and the leaves $B(m)$ of $\mathscr{B}$ lie in the orthogonal family of $(D-K+1)$-planes. But if ϕ is one-one, the fact that the ϕ-images of $A(m)$ and $B(m)$ intersect only once implies that $A(m)$ and $B(m)$ intersect only at m; thus M is isometric to $A(m) \times B(m)$, and ϕ is expressible as the product of an immersion of $A(m)$ in $\boldsymbol{R}^{K+1}$ and an immersion of $B(m)$ in $\boldsymbol{R}^{D-K+1}$. q.e.d.

In order to prove Theorem 8.2, we first give some lemmas which apply in both cases (i) and (ii), without restriction on the normal curvature tensor of ϕ. As above, M carries self-parallel distributions $\mathscr{A}$ and $\mathscr{B} = \mathscr{A}^{\perp}$ of dimensions K and $D-K$ respectively. The holonomy algebra h_m of M at m is the sum of the orthogonal algebras on $\mathscr{A}(m)$ and $\mathscr{B}(m)$.

8.4. Lemma. *The relative nullity space of $\phi | A(m)$ at m is the intersection of $\mathscr{A}(m)$ and the relative nullity space of ϕ.*

Proof. By (4), since $A(m)$ is totally geodesic in M, we need only show that the relative nullity space of $\phi | A(m)$ lies in the relative nullity space of ϕ. Furthermore, if x in $\mathscr{A}(m)$ lies in the relative nullity space of $\phi | A(m)$, then $T_x \mathscr{A}(m) = 0$, where T is the difference operator of ϕ. It remains to show $T_x \mathscr{B}(m) = 0$.

Suppose instead that $T_x y \neq 0$ for some y in $\mathscr{B}(m)$. Then

$$\langle R_{xy}x, y\rangle = \langle [T_x, T_y]x, y\rangle = -\langle T_x y, T_y x\rangle + \langle T_y y, T_x x\rangle = -|T_x y|^2 \neq 0,$$

where R is the curvature tensor of M. But it is an immediate consequence of the local product structure of M that $R_{xy} = 0$. q.e.d.

Now at each m in M, let $r(m)$ be the subalgebra of h_m which is generated by the curvature transformations of M at m. In bivector notation, we have $h_m = \mathscr{A}(m)^2 + \mathscr{B}(m)^2$, and, by Lemma 4.2, $r(m)$ generated by $\{\Sigma_a (S_a x)(S_a y) \mid x, y \text{ in } M_m\}$ where the z_a are a normal frame at m $(D+1 \leq a \leq D+2)$.

Let $D(S_a)$ be the subspace of M_m, which is simultaneously the range of the symmetric transformation S_a, the orthogonal complement of the kernel, and the span of the non-nullity eigenvectors. Then the following lemma is a consequence of [1], in which the possibilities for $r(m)$ are determined.

8.5. Lemma. *At any m in M there is a choice of normal frame for which each S_a has the property: if the rank of S_a is not one, then $D(S_a)$ lies in $\mathscr{A}(m)$ or in $\mathscr{B}(m)$.*

Proof. We observe that if x and y are independent vectors in M_m, and xy is in $\mathscr{A}(m)^2 + \mathscr{B}(m)^2$, then x and y are both in $\mathscr{A}(m)$ or both in $\mathscr{B}(m)$. Thus, if V^2 lies in $\mathscr{A}(m)^2 + \mathscr{B}(m)^2$, and the dimension of V is not one, then V lies in $\mathscr{A}(m)$ or in $\mathscr{B}(m)$.

Now if the normal frame can be chosen so that $D(S_{D+1}) \neq D(S_{D+2})$, then $D(S_{D+1})^2 + D(S_{D+2})^2$ generates $r(m)$ by Lemma 8 of [1], so that each $D(S_a)^2$ lies in $\mathscr{A}(m)^2 + \mathscr{B}(m)^2$.

Otherwise, there is a subspace V of M_m for which, for every choice of normal frame, $D(S_{D+1}) = D(S_{D+2}) = V$. By Theorem 1(b′) of [1], $r(m)$ is either V^2 or the unitary algebra of an isometric complex structure J on V. In the first case, V^2 lies in $\mathscr{A}(m)^2 + \mathscr{B}(m)^2$. In the second case, for an orthonormal basis $x_1, \cdots, x_{D'}, Jx_1, \cdots, Jx_{D'}$ of V, $\mathscr{A}(m)^2 + \mathscr{B}(m)^2$ contains all x_iJx_i and all $x_ix_j + Jx_iJx_j$; it follows from the initial remark that then V lies in $\mathscr{A}(m)$ or in $\mathscr{B}(m)$.

8.6. Corollary. *$T_{\mathscr{A}(m)}\mathscr{B}(m) = 0$ if and only if there is a choice of normal frame at m for which each $D(S_a)$ lies in $\mathscr{A}(m)$ or in $\mathscr{B}(m)$.*

Proof. Suppose $T_{\mathscr{A}(m)}\mathscr{B}(m) = 0$. Then $S_a\mathscr{A}(m) \subset \mathscr{A}(m)$, $S_a\mathscr{B}(m) \subset \mathscr{B}(m)$ for any S_a; in particular, if S_a has rank one, then its range is in $\mathscr{A}(m)$ or in $\mathscr{B}(m)$. Choose the normal frame as in Lemma 8.5.

8.7. Lemma. *Suppose that at every m in M, $T_{\mathscr{A}(m)}\mathscr{B}(m) = 0$. Then $\phi \circ \pi$ is the product of two Euclidean immersions of hypersurfaces.*

Proof. By hypothesis, $\mathscr{B}$ is Euclidean self-parallel on each leaf of $\mathscr{A}$, and $\mathscr{A}$, on each leaf of $\mathscr{B}$. We will show that the leaves of $\mathscr{A}$ lie in parallel Euclidean $(K + 1)$-planes under ϕ, hence that the leaves of $\mathscr{B}$ lie in the orthogonal family of $(D - K + 1)$-planes. By the de Rham product theorem, this suffices.

We use the index convention $1 \leq i, j \leq K$, $K + 1 \leq r, s \leq D$, $D + 1 \leq a, b \leq D + 2$. A frame field $X_1, \cdots, X_{D+2}$ on M, for which the X_i lie everywhere in $\mathscr{A}$ and the X_r lie in $\mathscr{B}$, will be called "adapted". Then we have $\nabla_{X_i}X_r = \bar{\nabla}_{X_i}X_r$ and $\nabla_{X_r}X_i = \bar{\nabla}_{X_r}X_i$.

Suppose $N \subset A(m)$ is an open subset carrying an adapted frame field for which each X_r is Euclidean self-parallel. Since M is locally a Riemannian product, this field may be extended locally to an adapted frame field satisfying $\nabla_{X_i}X_r = \nabla_{X_r}X_i = 0$ on an open subset of M. For such an extension, we have

$$\begin{aligned} 0 &= \langle \bar{\nabla}_{[X_i, X_r]} X_s, X_a \rangle - \langle [\bar{\nabla}_{X_i}, \bar{\nabla}_{X_r}] X_s, X_a \rangle \\ &= - \langle \bar{\nabla}_{X_i} T_{X_r} X_s, X_a \rangle . \end{aligned}$$

Thus on N we have

$$(7) \qquad \begin{aligned} 0 &= \langle T_{X_r} X_s, X_b \rangle \langle \bar{\nabla}_{X_i} X_b, X_a \rangle \\ &\quad + X_i \langle T_{X_r} X_s, X_a \rangle \qquad \text{for } b \neq a . \end{aligned}$$

Consider a point m in M, at which $T_{\mathscr{B}(m)}\mathscr{B}(m) \neq 0$; such a point exists, since M is compact. There is a choice of normal frame at m satisfying $D(S_{D+1}) \subset \mathscr{A}(m)$ and $D(S_{D+2}) \subset \mathscr{B}(m)$. Indeed, suppose not; then both $D(S_a)$ must lie in $\mathscr{B}(m)$ by Corollary 8.6, and the corresponding second fundamental form transformations S_a^B of $\phi \mid B(m)$ at m are not scalar multiples (otherwise a rotation of the normal frame would send one S_a into zero, and $D(0) \subset \mathscr{A}(m)$). Let C be the connected component of m in the subset of $A(m)$ consisting of points at which the S_a^B are not scalar multiples (for some choice of normal frame and hence for every choice). C is open in $A(m)$ and contains only points n satisfying $T_{\mathscr{A}(n)}M_n = 0$. We may therefore choose a Euclidean self-parallel, adapted frame field $X_1, \cdots, X_{D+2}$ on C. By (7), the corresponding functions $-\langle T_{X_r}X_s, X_a\rangle = \langle S_a X_r, X_s \rangle$ are constant on C. It follows that $C = A(m)$ is a complete K-plane under ϕ, in contradiction to the compactness of M.

Now, at the given point m, choose a neighbourhood N in $A(m)$ as described above (see (7)), with frame field $X_1, \cdots, X_{D+2}$; take N sufficiently small so that $T_{\mathscr{B}(n)}\mathscr{B}(n) \neq 0$ everywhere. It is easily shown possible to assume a smooth choice of the X_a satisfying $D(S_{D+1}) \subset \mathscr{A}(n)$ and $D(S_{D+2}) \subset \mathscr{B}(n)$ at every n in N. (The line $L_a(n)$ in which $X_a(n)$ may be chosen is of course uniquely determined.) By (7), since $\langle T_{X_r}X_s, X_{D+1}\rangle = 0$ and $T_{\mathscr{B}(n)}\mathscr{B}(n) \neq 0$, we have $\langle \bar{\nabla}_{X_i} X_{D+2}, X_{D+1} \rangle = 0$. Since also $\langle \bar{\nabla}_{X_i} X_{D+2}, X_r \rangle = - \langle \bar{\nabla}_{X_i} X_r, X_{D+2} \rangle = 0$ and $\langle \bar{\nabla}_{X_i} X_{D+2}, X_j \rangle = 0$, X_{D+2} is Euclidean self-parallel on N. It follows from (7) that the $T_{X_r}X_s$ are Euclidean self-parallelel on N.

It may be concluded that $T_{\mathscr{B}(n)}\mathscr{B}(n) \neq 0$ for all n in $A(m)$. And since then the $L_{D+2}(n)$ and $\mathscr{B}$ span on $A(m)$ a Euclidean self-parallel distribution orthogonal to $\mathscr{A}$, $A(m)$ lies in a Euclidean $(K+1)$-plane P under ϕ.

For any point m' in M, $B(m')$ intersects $A(m)$. Since $\mathscr{A}$ is Euclidean self-parallel on $B(m')$, $\mathscr{A}(m')$ is parallel in $\boldsymbol{R}^{D+2}$ to P. It follows that ϕ sends every leaf of $\mathscr{A}$ into a $(K+1)$-plane parallel to P.

8.8. Lemma. *Suppose $T_{\mathscr{A}(m)}\mathscr{B}(m) = 0$ at every point m at which M has non-zero curvature. Then $T_{\mathscr{A}(m)}\mathscr{B}(m) = 0$ everywhere.*

Proof. We may assume $K \geq D - K$. Then $K \geq 2$.

Consider the open subset C of M consisting of points at which $T_{\mathscr{A}(m)}\mathscr{B}(m) \neq 0$, and suppose C to be not empty. Let the relative nullity index ν of ϕ

take its minimum for C on the open subset N of C, and let $\mathscr{R}$ be the relative nullity distribution on N.

We have $\nu(N) \geq D - 2 > 0$. Indeed, $r(m) = 0$ at any m in C by assumption. By the proof of Lemma 8.5, there is a choice of normal frame at m for which each $D(S_a)^2$ is zero, hence for which each S_a has rank zero or one. Since $\mathscr{R}(m)$ is just the orthogonal complement of $D(S_{D+1}) + D(S_{D+2})$, $\nu(m) \geq D - 2$. (Note: This conclusion is of course already available in [3], where it is proved that for a Euclidean immersion of codimension E, $r(m) = 0$ implies $\nu(m) \geq D - E$.)

Consider a geodesic ray $\gamma\colon [0, \infty) \to M$ starting at m in N with initial velocity vector in $\mathscr{R}(m) - \{0\}$. If $\gamma[0, c)$ lies in N, then $\gamma[0, c)$ is a geodesic segment in the leaf through m of $\mathscr{R}$. It follows from Theorem 6.2 that γ leaves N when it leaves C: say, $\gamma[0, c) \subset N$ and $\gamma(c) = m^* \notin C$. Here, the compactness of M implies that γ is not complete in N. Then we have $\mathscr{A}(m^*) = \mathscr{A}(m)^*$ and $\mathscr{B}(m^*) = \mathscr{B}(m)^*$ (by Lemma 5.1), and the relative nullity space of ϕ at m^* equal to $\mathscr{R}(m)^*$. Since $T_{\mathscr{A}(m^*)}\mathscr{B}(m^*) = 0$, Corollary 8.6 implies that $\mathscr{R}(m)^*$ is spanned by vectors lying in $\mathscr{A}(m^*)$ and $\mathscr{B}(m^*)$. It follows that $\mathscr{R}(m)^*$ intersects $\mathscr{A}(m^*)$; otherwise, by considering dimensions, $\mathscr{R}(m)^* = \mathscr{B}(m^*) = \mathscr{B}(m)^*$ and $T_{\mathscr{B}(m)} = 0$, in contradiction to the choice of m.

We conclude that at any m in N, $\mathscr{R}(m)$ intersects $\mathscr{A}(m)$ non-trivially. Thus if n is a given point of N, the relative nullity of $\phi | A(n)$ is non-zero everywhere in $A(n) \cap N$ by Lemma 8.4. Let $\bar{N}$ be the subset of $A(n) \cap N$ on which the relative nullity of $\phi | A(n)$ is minimal, and let $\bar{\mathscr{R}} = \mathscr{R} \cap \mathscr{A}$ be the corresponding distribution on $\bar{N}$. A geodesic ray γ in $A(n)$, starting at m in $\bar{N}$ with initial velocity vector in $\bar{\mathscr{R}}(m) - \{0\}$, will lie in the leaf through m of $\bar{\mathscr{R}}$ until leaving N, and hence until leaving C: say, at $\gamma(c) = m^* \notin C$. But then $T_{\mathscr{A}(m^*)}\mathscr{B}(m^*) = 0$ and $T_{\mathscr{A}(m)}\mathscr{B}(m) \neq 0$, and by Lemma 6.3 that is impossible.

Proof of Theorem 8.2. We need only show that $T_{\mathscr{A}(m)}\mathscr{B}(m) = 0$ at every m in M at which $r(m) \neq 0$:

Case (i). $2 \leq K \leq D - 2$.

Suppose $T_{\mathscr{A}(n)}\mathscr{B}(n) \neq 0$ and, say, $r(n) \cap \mathscr{B}(n)^2 \neq 0$. Let N be the open subset of $A(n)$ consisting of all points there for which $T_{\mathscr{A}(m)}\mathscr{B}(m) \neq 0$. By the local product structure of M, we also have $r(m) \cap \mathscr{B}(m)^2 \neq 0$ for every m in N. Then it follows from Corollary 8.6 and Lemma 8.5 that at each m in N we may choose a normal frame for which $D(S_{D+2})$ lies in $\mathscr{B}(m)$ and S_{D+1} has rank one, so that the relative nullity of $\phi | A(n)$ is $K - 1 > 0$ on N. But then Lemma 6.3 and the compactness of M again imply that no such original point n exists.

Case (ii). $K = 1$ or $D - 1$; ϕ has zero normal curvature everywhere.

Suppose $K = D - 1$. Let C be the open subset of M consisting of all points m at which $T_{\mathscr{A}(m)}\mathscr{B}(m) \neq 0$ and $r(m) \neq 0$. Suppose C to be not empty.

At any m in C, there is a normal frame, for which $D(S_{D+1})$ lies in $\mathscr{A}(m)$ and has dimension at least two (since $0 \neq r(m) \subset \mathscr{A}(m)^2$), and S_{D+2} has rank one. Further, if y is a non-nullity eigenvector of S_{D+2}, then $y = cy_1 + dy_2$, where y_1 and y_2 are unit vectors in $\mathscr{A}(m)$ and $\mathscr{B}(m)$ respectively, and $cd \neq 0$. Since the S_a are simultaneously diagonalizable by Lemma 4.1, and y is not in $D(S_{D+1})$, $S_{D+1}y = 0$. Then $S_{D+1}y_1 = 0$ and $S_{D+1}(-dy_1 + cy_2) = 0$. Since also $S_{D+2}(-dy_1 + cy_2) = 0$, the relative nullity index ν of ϕ is positive at m.

Let N be the open subset of C, on which ν takes its minimum for C, and let $\mathscr{R}$ be the relative nullity distribution on N. A geodesic ray, starting at m in N with initial velocity vector in $\mathscr{R}(m) - \{0\}$, lies in a leaf of $\mathscr{R}$ until leaving C at a point m^*. Then $\nu(m) = \nu(m^*)$, and the relative nullity space of ϕ at m^* is the Euclidean parallel translate $\mathscr{R}(m)^*$ of $\mathscr{R}(m)$ to m^*. Finally, for a vector $y_2 \neq 0$ in $\mathscr{B}(m)$, $T_{y_2} \neq 0$ implies $T_{y_2*} \neq 0$.

Since $\nu(m^*) = \nu(m) \leq D - 3$, we must have $r(m^*) \neq 0$. Since m^* is not in C, it follows that $T_{\mathscr{A}(m^*)}\mathscr{B}(m^*) = 0$. By Corollary 8.6, since $T_{y_2*} \neq 0$, some $D(S_a)$ equals $\mathscr{B}(m^*)$, and the relative nullity space of ϕ at m^* lies in $\mathscr{A}(m^*)$, that is, $\mathscr{R}(m)^*$ lies in $\mathscr{A}(m)^*$. This is impossible, since $\mathscr{R}(m)$ is not in $\mathscr{A}(m)$. q.e.d.

The assumption of zero normal curvature in Theorem 8.2 (ii) cannot be omitted; Y. H. Clifton has given an example, for any $D > 1$, of a compact D-dimensional manifold M reducible with holonomy algebra $o(D-1)$ and having an isometric imbedding in $\boldsymbol{R}^{D+2}$, which is not a product imbedding.

It is a corollary to Theorem 8.2, that if an isometric immersion $\phi: M \to \boldsymbol{R}^{D+2}$ (M is compact and of dimension $D > 2$) has non-zero normal holonomy, then M is either irreducible or reducible to $o(D-1)$. The example cited above shows that the latter situation can occur.

9. A cylindricity theorem

Let ϕ be an isometric immersion of the complete D-dimensional manifold M in $\boldsymbol{R}^{D+E}$.

ϕ is said to be $(D-K)$-cylindrical if M and ϕ can be expressed as Riemannian products $M = M_1 \times \boldsymbol{R}^{D-K}$ and $\phi = \phi_1 \times \iota$, where ϕ_1 is an immersion of M_1 in $\boldsymbol{R}^{K+E}$, and ι is the identity map of $\boldsymbol{R}^{D-K}$. ϕ is $(D-K)$-cylindrical if and only if M carries a $(D-K)$-dimensional, Euclidean self-parallel distribution (that is, a self-parallel distribution $\mathscr{B}$ on M, which satisfies $T_{\mathscr{B}(m)} = 0$ everywhere). Indeed, if $\mathscr{B}$ is such a distribution, its leaves are complete parallel $(D-K)$-planes under ϕ. The leaves of $\mathscr{B}^\perp$ then lie in the orthogonal family of $(K+E)$-planes, and have the unique intersection property with the leaves of $\mathscr{B}$ since ϕ is one-one on each leaf of $\mathscr{B}$.

Certainly, then, if (M^*, π) is the simply connected Riemannian covering of M, and the immersion $\phi \circ \pi$ of M^* in $\boldsymbol{R}^{D+E}$ is $(D-K)$-cylindrical, then ϕ is also $(D-K)$-cylindrical.

Now let $I = I(M)$ be the number of non-trivial factors in the restricted holonomy group of M, that is, suppose the simply connected Riemannian covering manifold M^* has de Rham decomposition

$$M^* = M_1 \times \cdots \times M_I \times R^{D-\Sigma K_i},$$

where $K_i \geq 2$ is the dimension of the irreducible factor M_i.

9.1. Theorem. *Let M be a complete D-dimensional manifold, and ϕ an isometric immersion of M in R^{D+E} having zero normal curvature tensor. Then $I(M) \leq E$. If $I(M) = E$, then ϕ is $(D - \Sigma K_i)$-cylindrical.*

For codimensions $E = 1, 2$, the assumption of zero normal curvature need not be made.

Of course, every immersion of codimension one has zero normal curvature. Before giving the proof of Theorem 9.1, we also state the following theorem of Hartman and Louis Nirenberg:

9.2. Theorem [6]. *An isometric immersion ϕ in R^{D+1} of a flat, complete D-dimensional manifold M is $(D - 1)$-cylindrical.*

Proof [10]. Since M is flat (that is, has zero curvature tensor), and we may assume M simply connected, we have M isometric to R^D. The relative nullity of ϕ is $D - 1$ or D on M. Then Theorem 6.2 and the fact that complete non-intersecting $(D - 1)$-planes in R^D are parallel guarantee the existence of a self-parallel $(D - 1)$-dimensional distribution $\mathscr{B}$ on M satisfying $T_{\mathscr{B}(m)} = 0$ at every point. q.e.d.

We combine Theorem 9.1 for $E = 1$ and Theorem 9.2:

9.3. Corollary. *Let M be a complete D-dimensional manifold having an isometric immersion ϕ in R^{D+1}. Then $M = M_1 \times R^{D-K}$, where M_1 is irreducible, and ϕ is $(D - K)$-cylindrical.*

When M is not flat, the integer K in Corollary 9.3 is the dimension of the subspace of a tangent space M_m, which is spanned by the parallel translates to m of all $D(S_z)$, where S_z is a second fundamental form transformation of rank at least two.

Richard Sacksteder proved in [12] that if every sectional curvature of M is non-negative and at least one is positive, then Corollary 9.3 holds and K is in fact the maximal rank of the second fundamental form transformations of M. In this case, $\phi_1(M_1)$ was proved to be the boundary of a convex body, which contains no line, in R^{K+1}.

The result Corollary 9.3 was remarked by Simone Dolbeault-Lemoine [8] in the special case that M has no flat open submanifolds.

The proof of Theorem 9.1 requires the following algebraic lemma obtained by a method of [1]:

9.4. Lemma. *Suppose ϕ: $M \to R^{D+E}$ has zero normal curvature tensor. At every m in M, there is a choice of normal frame for which $\Sigma_{D+1 \leq a \leq D+E} D(S_a)^2$ generates $r(m)$.*

Proof. By Lemma 4.2, we need only find a normal frame at m, for which each $D(S_a)^2$ lies in $r(m)$.

By Lemma 4.1, there is an orthogonal basis $x_1, \cdots, x_D$ of M_m, for which $T_{x_i}x_j = 0$ whenever $i \neq j$. Then $T_{M_m}M_m$ is a subspace of $M_m{}^\perp$ and we may suppose $T_{x_1}x_1, \cdots, T_{x_{D'}}x_{D'}$ are a basis. Choose the normal frame $z_{D+1}, \cdots, z_{D+E}$ so that $T_{x_i}x_i$ is a linear combination of $z_{D+1}, \cdots, z_{D+i}$ $(1 \leq i \leq D')$.

If $D' > 0$, we have $S_{D+1}x_1 \neq 0$ and $S_a x_1 = 0$ for $a > D + 1$. By Lemma 4.2, $r(m)$ contains $(S_{D+1}x_1)(S_{D+1}x)$ for all x in M_m, and then by (5), $r(m)$ contains $D(S_{D+1})^2$. If $D' = 1$, then $S_a = 0$ for all $a > D + 1$. If $D' > 1$, then we have $S_{D+2}x_2 \neq 0$ and $S_a x_2 = 0$ for all $a > D + 2$, and $r(m)$ contains all $(S_{D+1}x_2)(S_{D+1}x) + (S_{D+2}x_2)(S_{D+2}x)$ and hence $D(S_{D+2})^2$, etc.

Proof of Theorem 9.1. By Lemma 9.4 and Lemma 8.5, all Euclidean isometric immersions with zero normal curvature tensor and all of codimension two have the property: at any m in M, if $r(m)$ lies in ΣU_i^2 (where the U_i are orthogonal subspaces of M_m), then there is a choice of normal frame for which each $D(S_a)$ lies in one of the U_i or else has dimension one.

The conclusion $I(M) \leq E$ of Lemma 9.1 follows. Indeed, there is a point m in M, at which $r(m)$ lying in an algebra $\Sigma_{1 \leq i \leq I(M)} U_i^2$ has non-trivial intersection with each U_i^2. Then Lemma 4.2 and the above remark imply that $I(M)$ does not exceed E.

Now, given an immersion $\phi\colon M \to \boldsymbol{R}^{D+E}$ with the property just discussed and a simply connected and complete M with de Rham decomposition $M_1 \times \cdots \times M_E \times \boldsymbol{R}^{D-\Sigma K_i}$, we must show ϕ to be $(D - \Sigma K_i)$-cylindrical. Thus if $\mathscr{A}_1, \cdots, \mathscr{A}_E, \mathscr{B}$ are the self-parallel distributions on M corresponding to the given product structure, we must prove that $T_{\mathscr{B}(m)} = 0$ everywhere, that is, that $\mathscr{B}(m)$ lies in the relative nullity space of ϕ at every m in M.

Let C be the open subset of M consisting of all points n at which the relative nullity space of ϕ does not contain $\mathscr{B}(n)$, and suppose C to be not empty. Observe that if $r(m)$, which lies in $\Sigma_{1 \leq i \leq E}\mathscr{A}_i(m)^2$, intersects each of the $\mathscr{A}_i(m)^2$ non-trivially, then m is not in C. Choose a point n of C, at which $r(n)$ non-trivially intersects a maximal number (for C) of the $\mathscr{A}_i(n)^2$, say $\mathscr{A}_{I'+1}(n), \cdots, \mathscr{A}_E(n)$ where $1 \leq I' \leq E$. Let L be the leaf through n of $\mathscr{A}_1 + \cdots + \mathscr{A}_{I'} + \mathscr{B}$.

By the argument of Lemma 8.4, at any point m in L the relative nullity space of $\phi | L$ is the intersection of $\mathscr{A}_1(m) + \cdots + \mathscr{A}_{I'}(m) + \mathscr{B}(m)$ and the relative nullity space of ϕ. Thus $L \cap C$ contains exactly those points m of L, at which the relative nullity space of $\phi | L$ does not contain $\mathscr{B}(m)$.

The choice of L ensures that $L \cap C$ contains only flat points of L. From an examination of the second fundamental forms of ϕ at a point of $L \cap C$ it follows that we may choose a set of second fundamental form transformations there for $\phi | L$, for which at most I' transformations are non-zero and of rank one. Thus if D' is the dimension of L, then the relative nullity index ν of $\phi | L$ takes values not less than $D' - I' > 0$ on $L \cap C$. Let N be the open subset of

$L \cap C$, on which ν takes its minimum for $L \cap C$, and let $\mathscr{R}$ be the corresponding relative nullity distribution on N.

Now suppose a geodesic ray γ in L, starting at m in N with initial velocity vector in $\mathscr{R}(m) - \{0\}$, leaves N, then by Theorem 6.2 we have $\gamma[0, c) \subset N$, $\gamma(c) = m^* \notin C$, and the relative nullity space of $\phi | L$ at m^* equal to the Euclidean parallel translate $\mathscr{R}(m)^*$ of $\mathscr{R}(m)$. Then $\mathscr{R}(m)^*$ contains $\mathscr{B}(m^*) = \mathscr{B}(m)^*$, and $\mathscr{R}(m)$ contains $\mathscr{B}(m)$, in contradiction to the choice of m in C. Thus we conclude that the leaves of $\mathscr{R}$ are complete (hence are complete Euclidean $\nu(N)$-planes under ϕ).

We choose a Euclidean self-parallel vector field X on one of the leaves of $\mathscr{R}$ such that $X(m)$ lies in $\mathscr{B}(m)$ and not in $\mathscr{R}(m)$ at some and hence every point m of the leaf. At each m, the geodesic γ_m tangent to $X(m)$ in L lies entirely in a leaf of $\mathscr{B}$ and contains only flat points of L (since m is a flat point of L). The union of the γ_m is easily seen to lie in a flat open subset of L, and hence to form the image of $\boldsymbol{R}^{\nu(N)+1} = P$ under a totally geodesic isometric immersion φ of P in L.

Since $\nu(N) + 1$ exceeds $D' - I'$, then at p in P, the orthogonal projection of $d\varphi(P_p)$ into $\mathscr{A}_i(\varphi(p))$ is onto for some i $(1 \leq i \leq I')$. Since φ is totally geodesic, and the geodesics of a product are products of geodesics, the projection mapping of $\varphi(P)$ into any leaf of $\mathscr{A}_i$ is onto; this is impossible, since $\varphi(P)$ contains only flat points of L.

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